Bridgeland Stability of Line Bundles on Smooth Projective Surfaces

Saint Vincent College

Daniele Arcara

daniele.arcara@email.stvincent.edu

Colorado State University

Eric Miles

miles@math.colostate.edu

Overview

Idea

A **Bridgeland Stability Condition** (**BSC**) on a smooth projective variety X assigns the label of **(semi)stable** or **unstable** to each complex $E \in D^b(\mathrm{Coh}\, X) =: D(X)$ in a meaningful way.

Question

When is $\mathcal{O}_S$ given the label semistable in a certain class of "divisorial" BSCs defined on surfaces?
(this will characterize the stability of all line bundles)

Theorem 1

If the surface S has no curves C satisfying $C^2 < 0$, then $\mathcal{O}_S$ is σ-stable for *all* $\sigma \in \mathrm{Stab}_{div}(S)$

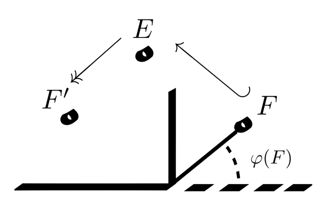
Bridgeland Stability Conditions

General Setup

Let S be a smooth projective surface. A **stability condition** σ is a pair $\sigma = (Z, \mathcal{A})$ where…

- $\mathcal{A}$ is a *heart* of $D(S)$
- $Z : K_{num}(S) \to \mathbb{C}$ is a group homomorphism

satisfying three properties:



1 (Positivity) for all $0 \neq E \in \mathcal{A}$, have $Z(E) \in \{re^{i\pi} \mid r > 0,\ 0 < \varphi \leq 1\}$.

We say $E \in \mathcal{A}$ is σ-**(semi)stable** if for all nontrivial $F \hookrightarrow E$ in $\mathcal{A}$ we have $\varphi(E) > (\geq)\ \varphi(F)$.

2 (HN-Filtrations) objects $E \in \mathcal{A}$ have well-behaved filtrations in terms of σ-semistable objects

3 (Support Property) images of object classes via Z do not accumulate at the origin

$\mathrm{Stab}(X) = \{\text{all BSCs on } X\}$ is a complex manifold [3].

Divisorial BSCs

By [2], a choice of ample $\mathbb{R}$-divisor H and general $\mathbb{R}$-divisor D give a BSC $\sigma_{D,H}$ where

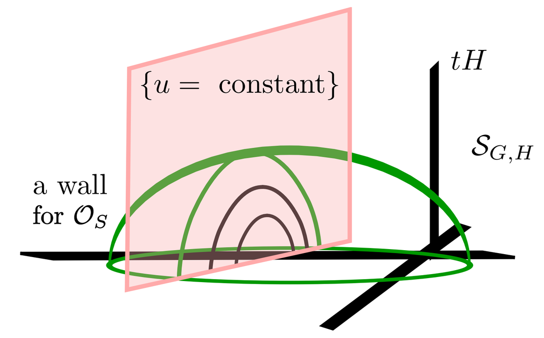
- $\mathcal{A}_{D,H}$ is the tilt of $\mathrm{Coh}\, X$ at the Mumford H-slope $D.H$
- $Z_{D,H}(E) = -\int e^{-(D+iH)}\mathrm{ch}(E)$

We denote the set of all such BSCs by $\mathrm{Stab}_{div}(S)$.

Walls in $\mathrm{Stab}_{div}(S)$

Slices of $\mathrm{Stab}_{div}(S)$

A choice of ample divisor H and divisor G such that $H.G = 0$ give us a 3-space $\mathcal{S}_{G,H} \subset \mathrm{Stab}_{div}(S)$ containing the stability conditions $\sigma_{D,A}$ where $D = sH + uG$ and $A = tH$ for $s, u \in \mathbb{R}$ and $t > 0$.



- In $\mathcal{S}_{G,H}$, the **walls** for $\mathcal{O}_S$ (i.e. the set of BSCs σ with some $E \hookrightarrow \mathcal{O}_S$ and $\varphi(E) = \varphi(\mathcal{O}_S)$) are quadric surfaces
- inside the planes $\{u = \text{constant}\}$, the walls for $\mathcal{O}_S$ are nested [4], so we may consider just the $t = 0$-plane to understand the position of walls for $\mathcal{O}_S$.

When does $E \hookrightarrow \mathcal{O}_S$?

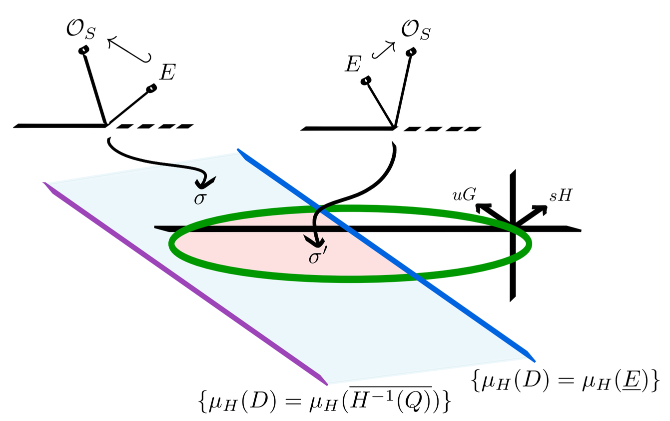
- $E \hookrightarrow \mathcal{O}_S$ in $\mathcal{A}_{D,H}$ implies E is a sheaf, but $Q := \mathrm{coker}\,(E \hookrightarrow \mathcal{O}_S)$ may be a two-term complex $Q = Q_{-1} \to Q_0$.

There is a HN-filtration of Mumford H-semistable sheaves for both E and $H^{-1}(Q)$.

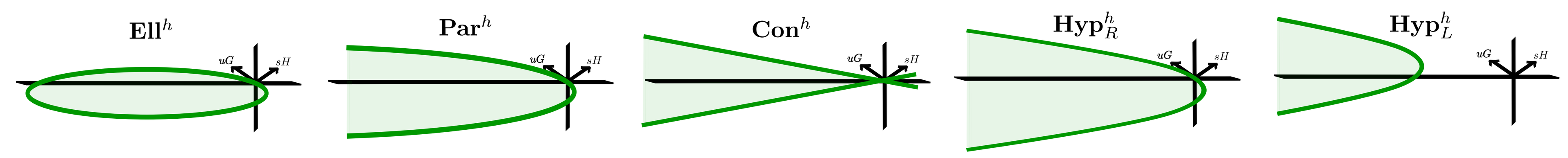
- For $E \hookrightarrow \mathcal{O}_S$ at $\sigma_{D,tH}$, we require…

$$\mu_H(\overline{H^{-1}(Q)}) \leq \mu_H(D) < \mu_H(\underline{E})$$

- where μ_H denotes the Mumford H-slope
- $\overline{H^{-1}(Q)}$ and $\underline{E}$ denote respectively the Mumford H-semistable factor of $H^{-1}(Q)$ (of E) with the largest (smallest) H-slope



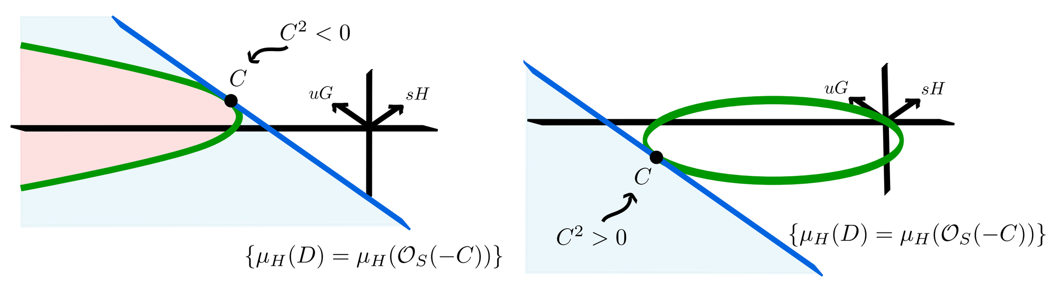
Wall Types for $\mathcal{O}_S$ in $t = 0$-plane



Tools for Proof

Rank 1 Case

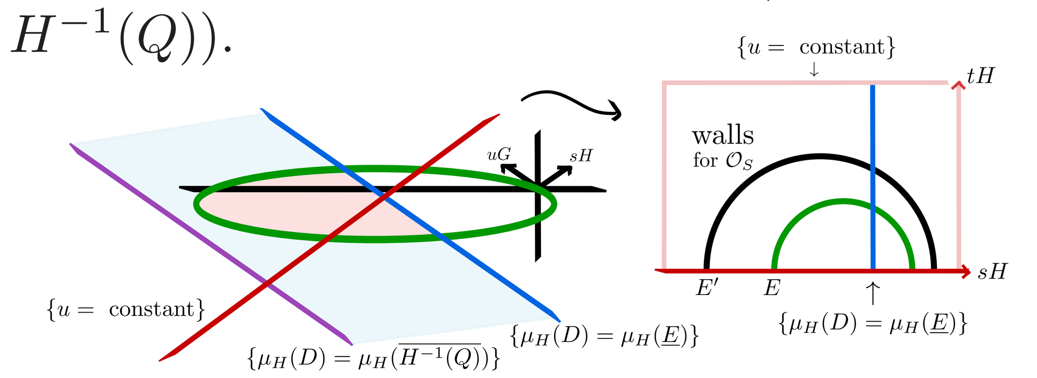
Let C be the class of a curve. The wall for $\mathcal{O}_S(-C) \hookrightarrow \mathcal{O}_S$ is determined by the point in the $t = 0$-plane corresponding to C.



The above pictures show that $\mathcal{O}_S(-C) \hookrightarrow \mathcal{O}_S$ and $\varphi(\mathcal{O}_S(-C)) \geq \varphi(\mathcal{O}_S)$ implies that $C^2 < 0$, i.e. that C is a **negative curve**.

Bertram's Lemma

In [1] it is shown that in a plane $\{u = \text{constant}\}$ where the wall for $E \hookrightarrow \mathcal{O}_S$ intersects the line $\{\mu_H(D) = \mu_H(\underline{E})\}$ or the line $\{\mu_H(D) = \mu_H(\overline{H^{-1}(Q)})\}$, there is a wall above the one for E which is obtained by omitting $\underline{E}$ (respectively $\overline{H^{-1}(Q)}$) from the Mumford $H-$filtration of E (respectively $H^{-1}(Q)$).

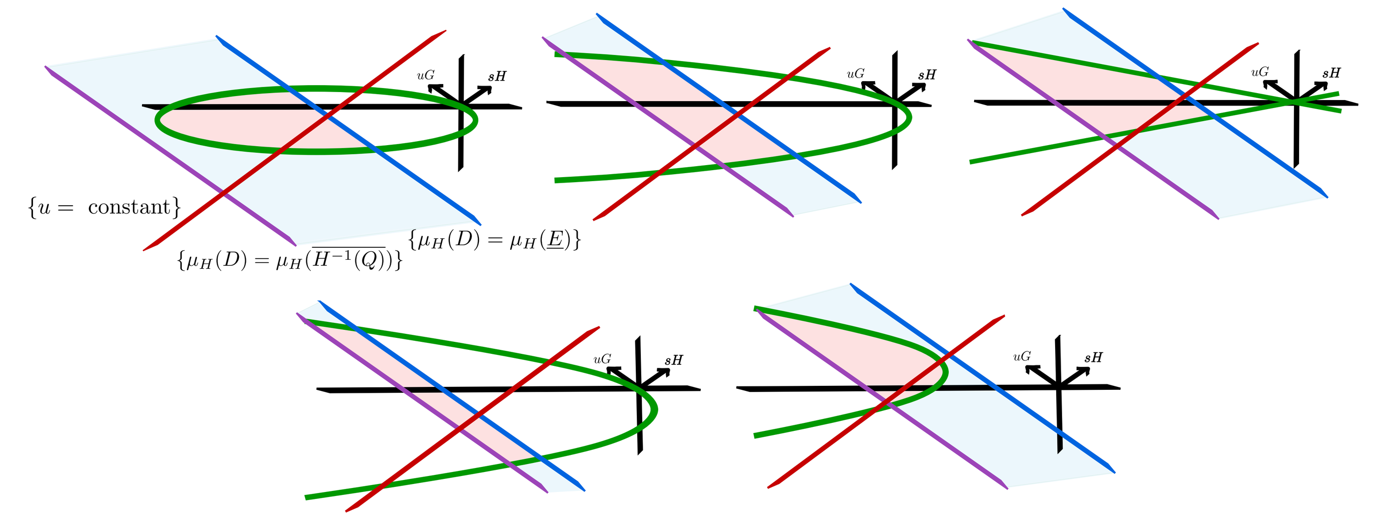


Sketch of Proof of Theorem 1

The proof is by contradiction via minimal counterexample.

Since S has no curves C satisfying $C^2 < 0$, the Rank 1 Case shows that no rank 1 E can destabilize $\mathcal{O}_S$. Now, suppose that at some σ there is an E with rank at least 2 satisfying $E \hookrightarrow \mathcal{O}_S$ and $\varphi(E) \geq \varphi(\mathcal{O}_S)$. We may suppose that no E' of lower rank satisfies this.

But, if we consider the wall types possible for $E \hookrightarrow \mathcal{O}_S$, we see one can always find a u such that Bertram's Lemma gives us an E' of lower rank than E which satisfies $E' \hookrightarrow \mathcal{O}_S$ and $\varphi(E') \geq \varphi(\mathcal{O}_S)$ at some σ' (see figure). This contradicts our assumption on E, and we are done.



Other Work

Theorem 2

If S has Picard rank 2 and has one irreducible negative curve C, then $\mathcal{O}_S$ is destabilized only by $\mathcal{O}_S(-C)$.

Future Work & Interests:

- S with Picard rank ≥ 2, general case
- Quiver regions for Del Pezzo surfaces
- Stability and birational geometry for 0-dim'l ideal sheaves

References

[1] D. Arcara, A. Bertram, I. Coskun, and J. Huizenga. "The Minimal Model Program for the Hilbert Scheme of Points on $\mathbb{P}^2$ and Bridgeland Stability". *Adv. Math.*, 235:580-626, 2013.

[2] D. Arcara and A. Bertram. "Bridgeland-Stable Moduli Spaces for K-Trivial Surfaces". *JEMS*, 15(1):1-38, 2013. (appendix by Max Lieblich).

[3] T. Bridgeland. "Stability Conditions on Triangulated Cateogories". *Ann. Math.*, 166:317-345, 2007.

[4] A. Maciocia. "Computing the Walls Associated to Bridgeland Stability Conditions on Projective Surfaces". `http://arxiv.org/abs/1202.4587`